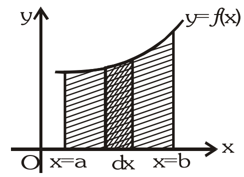
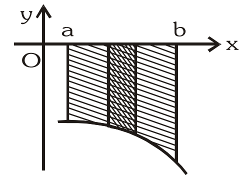
AREA UNDER THE CURVE

1. AREA UNDER THE CURVES :

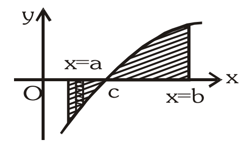
(a) Area bounded by the curve $y = f(x)$, the x-axis and the ordinates at $x = a$ and $x = b$ is given by $A = \int_a^b y\,dx$, where $y = f(x)$ lies above the x-axis and $b > a$. Here vertical strip of thickness dx is considered at distance x.



(b) If $y = f(x)$ lies completely below the x-axis then A is negative and we consider the magnitude only, i.e. $A = \left|\int_a^b y\,dx\right|$



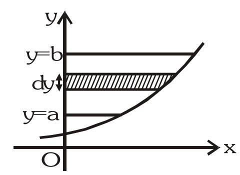
(c) If curve crosses the x-axis at $x = c$, then $A = \left|\int_a^c y\,dx\right| + \int_c^b y\,dx$



(d) Sometimes integration w.r.t. y is very useful (horizontal strip) :

Area bounded by the curve, y-axis and the two abscissae at

$y = a$ & $y = b$ is written as $A = \int_a^b x\,dy$.



Note : If the curve is symmetric and suppose it has 'n' symmetric portions, then total area = n (Area of one symmetric portion).

Illustration 1 : Find the area bounded by $y = \sec^2 x$, $x = \frac{\pi}{6}$, $x = \frac{\pi}{3}$ & x-axis

Solution : Area bounded $= \int_{\pi/6}^{\pi/3} y\,dx = \int_{\pi/6}^{\pi/3} \sec^2 x\,dx = [\tan x]_{\pi/6}^{\pi/3} = \tan\frac{\pi}{3} - \tan\frac{\pi}{6} = \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$ sq.units.

Illustration 2 : Find the area in the first quadrant bounded by $y = 4x^2$, $x = 0$, $y = 1$ and $y = 4$.

Solution : Required area $= \int_1^4 x\,dy = \int_1^4 \frac{\sqrt{y}}{2}\,dy = \frac{1}{2}\left[\frac{2}{3}y^{3/2}\right]_1^4$

$= \frac{1}{3}[4^{3/2} - 1] = \frac{1}{3}[8 - 1]$

$= \frac{7}{3} = 2\frac{1}{3}$ sq.units.

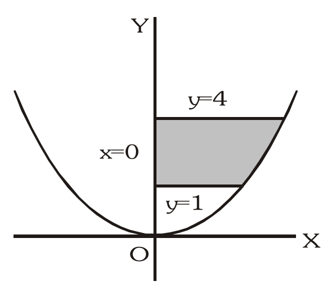
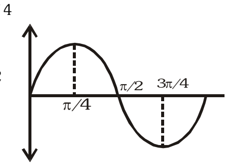


Illustration 3 : Find the area bounded by the curve $y = \sin 2x$, x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$

Solution : Required area $= \int_{\pi/4}^{\pi/2} \sin 2x\,dx + \left|\int_{\pi/2}^{3\pi/4} \sin 2x\,dx\right| = \left(-\frac{\cos 2x}{2}\right)\Bigg|_{\pi/4}^{\pi/2} + \left|\left(-\frac{\cos 2x}{2}\right)\right|_{\pi/2}^{3\pi/4}$

$= -\frac{1}{2}[-1-0] + \left|\frac{1}{2}(0+(-1))\right| = 1$ sq. unit



Do yourself - 1 :

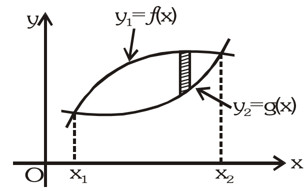
(i) Find the area bounded by $y = x^2 + 2$ above x-axis between $x = 2$ & $x = 3$.

(ii) Using integration, find the area of the curve $y = \sqrt{1-x^2}$ with co-ordinate axes bounded in first quadrant.

(iii) Find the area bounded by the curve $y = 2\cos x$ and the x-axis from $x = 0$ to $x = 2\pi$.

(iv) Find the area bounded by the curve $y = x|x|$, x-axis and the ordinates $x = -\frac{1}{2}$ and $x=1$.

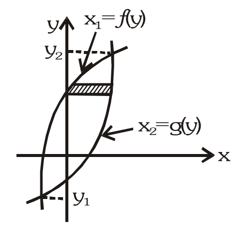
2. AREA ENCLOSED BETWEEN TWO CURVES :



(a) Area bounded by two curves $y = f(x)$ & $y = g(x)$ such that $f(x) > g(x)$ is

$$A = \int_{x_1}^{x_2} (y_1 - y_2)dy$$

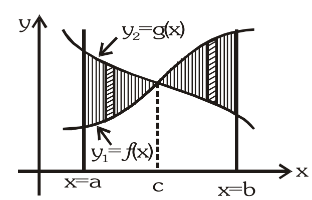
$$A = \int_{x_1}^{x_2} [f(x) - g(x)]dx$$



(b) In case horizontal strip is taken we have

$$A = \int_{y_1}^{y_2} (x_1 - x_2)dy$$

$$A = \int_{y_1}^{y_2} [f(y) - g(y)]dy$$



(c) If the curves $y_1 = f(x)$ and $y_2 = g(x)$ intersect at $x = c$, then required area

$$A = \int_a^c (g(x) - f(x))dx + \int_c^b (f(x) - g(x))dx = \int_a^b \left|f(x) - g(x)\right|dx$$

Note : Required area must have all the boundaries indicated in the problem.

Illustration 4 : Find the area bounded by the curve $y = (x - 1)(x - 2)(x - 3)$ lying between the ordinates $x = 0$ and $x = 3$ and x-axis

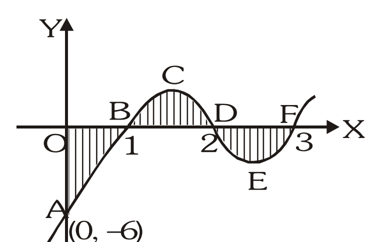
Solution : To determine the sign, we follow the usual rule of change of sign.

$y = +ve$ for $x > 3$

$y = -ve$ for $2 < x < 3$

$y = +ve$ for $1 < x < 2$

$y = -ve$ for $x < 1$.



$$\int_0^3 |y|\,dx = \int_0^1 |y|\,dx + \int_1^2 |y|\,dx + \int_2^3 |y|\,dx$$

$$= \int_0^1 -y\,dx + \int_1^2 y\,dx + \int_2^3 -y\,dx$$

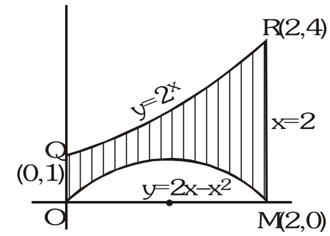
Now let $F(x) = \int (x-1)(x-2)(x-3)\,dx = \int (x^3 - 6x^2 + 11x - 6)\,dx = \frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 - 6x$.

$\therefore$ $F(0) = 0$, $F(1) = -\frac{9}{4}$, $F(2) = -2$, $F(3) = -\frac{9}{4}$.

Hence required Area $= -[F(1) - F(0)] + [F(2) - F(1)] - [F(3) - F(2)] = 2\frac{3}{4}$ sq.units.

Illustration 5 : Compute the area of the figure bounded by the straight lines $x = 0$, $x = 2$ and the curves $y = 2^x$, $y = 2x - x^2$.

Solution : Figure is self-explanatory $y = 2^x$, $(x - 1)^2 = -(y - 1)$



The required area $= \int_0^2 (y_1 - y_2)\,dx$

where $y_1 = 2^x$ and $y_2 = 2x - x^2$ $= \int_0^2 (2^x - 2x + x^2)dx$

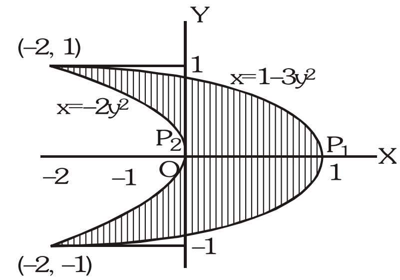
$$= \left[\frac{2^x}{\ln 2} - x^2 + \frac{1}{3}x^3\right]_0^2 = \left(\frac{4}{\ln 2} - 4 + \frac{8}{3}\right) - \frac{1}{\ln 2} = \frac{3}{\ln 2} - \frac{4}{3} \text{ sq.units.}$$

Illustration 6 : Compute the area of the figure bounded by the parabolas $x = -2y^2$, $x = 1 - 3y^2$.

Solution : Solving the equations $x = -2y^2$, $x = 1 - 3y^2$, we find that ordinates of the points of intersection of the two curves as $y_1 = -1$, $y_2 = 1$.

The points are $(-2, -1)$ and $(-2, 1)$.

The required area



$$2\int_0^1 (x_1 - x_2)\,dy = 2\int_0^1 [(1 - 3y^2) - (-2y^2)]dy$$

$$=2\int_0^1 (1 - y^2)dy = 2\left[y - \frac{y^3}{3}\right]_0^1 = \frac{4}{3} \text{ sq.units.}$$

Do yourself - 2 :

(i) Find the area bounded by $y = \sqrt{x}$ and $y = x$.

(ii) Find the area bounded by the curves $x = y^2$ and $x = 3 - 2y^2$.

(iii) Find the area of the region bounded by the curves $x = \frac{1}{2}$, $x = 2$, $y = \log x$ and $y = 2^x$.

3. CURVE TRACING :

The following procedure is to be applied in sketching the graph of a function $y = f(x)$ which in turn will be extremely useful to quickly and correctly evaluate the area under the curves.

(a) Symmetry : The symmetry of the curve is judged as follows :

  (i) If all the powers of y in the equation are even then the curve is symmetrical about the axis of x.

  (ii) If all the powers of x are even, the curve is symmetrical about the axis of y.

  (iii) If powers of x & y both are even, the curve is symmetrical about the axis of x as well as y.

  (iv) If the equation of the curve remains unchanged on interchanging x and y, then the curve is symmetrical about $y = x$.

  (v) If on interchanging the signs of x & y both, the equation of the curve is unaltered then there is symmetry in opposite quadrants.

(b) Find dy/dx & equate it to zero to find the points on the curve where you have horizontal tangents.

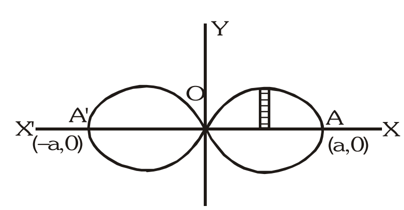
(c) Find the points where the curve crosses the x–axis & also the y–axis.

(d) Examine if possible the intervals when f(x) is increasing or decreasing. Examine what happens to 'y' when $x \to \infty$ or $-\infty$.

Illustration 7 : Find the area of a loop as well as the whole area of the curve $a^2y^2 = x^2(a^2 - x^2)$.

Solution : The curve is symmetrical about both the axes. It cuts x-axis at $(0, 0)$, $(-a, 0)$, $(a, 0)$

Area of a loop $= 2\int_0^a y\,dx = 2\int_0^a \frac{x}{a}\sqrt{a^2 - x^2}\,dx$



$$= -\frac{1}{a}\int_0^a \sqrt{a^2 - x^2}(-2x)dx = -\frac{1}{a}\left[\frac{2}{3}(a^2 - x^2)^{3/2}\right]_0^a = \frac{2}{3}a^2$$

Total area $= 2\left(\frac{2}{3}a^2\right) = \frac{4}{3}a^2$ sq.units.

Illustration 8 : Find the whole area induded between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes.

Solution : (i) The curve is symmetric about both the axes (even powers of x & y)

(ii) Asymptotes are $x = \pm a$

$$A = 4\int_0^a y\,dx$$

$$= 4\int_0^a \frac{ax}{\sqrt{a^2 - x^2}}dx$$

$$= 4a\left|-\sqrt{a^2 - x^2}\right|_0^a$$

$$= 4a^2$$

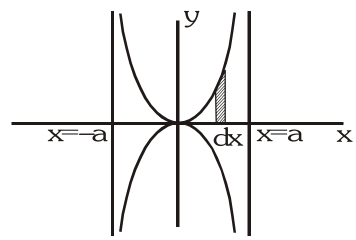
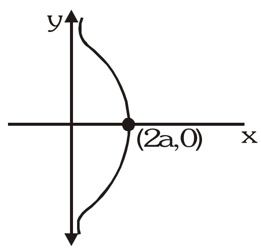


Illustration 9 : Find the area bounded by the curve $xy^2 = 4a^2(2a-x)$ and its asymptote.

Solution : (i) The curve is symmetrical about the x-axis as it contains even powers of y.

(ii) It passes through (2a,0).

(iii) Its asymptote is x = 0, i.e., y-axis.



$$A = 2\int_0^{2a} y\,dx = 2\int_0^{2a} 2a\sqrt{\frac{2a - x}{x}}dx$$

Put $x = 2a\sin^2\theta$

$$A = 16a^2\int_0^{\pi/2}\cos^2\theta\, d\theta$$

$$= 4\pi a^2$$

4. IMPORTANT POINTS :

(a) Since area remains invariant even if the co-ordinate axes are shifted, hence shifting of origin in many cases proves to be very convenient in computing the area.

Illustration 10 : Find the area enclosed by $|x - 1| + |y + 1| = 1$.

Solution : Shift the origin to (1, –1).

$X = x - 1$ $Y = y + 1$

$|X| + |Y| = 1$

Area $= \sqrt{2}\times\sqrt{2} = 2$ sq. units

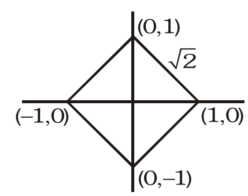


Illustration 11 : Find the area of the region common to the circle $x^2 + y^2 + 4x + 6y - 3 = 0$ and the parabola $x^2 + 4x = 6y + 14$.

Solution : Circle is $x^2 + y^2 + 4x + 6y - 3 = 0$

$\Rightarrow (x + 2)^2 + (y + 3)^2 = 16$

Shifting origin to (–2,–3).

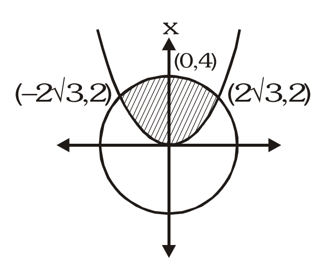
$X^2 + Y^2 = 16$

equation of parabola $\to (x + 2)^2 = 6(y + 3)$

$\Rightarrow X^2 = 6Y$

Solving circle & parabola, we get $X = \pm 2\sqrt{3}$

Hence they intersect at $\left(-2\sqrt{3}, 2\right)$ & $\left(2\sqrt{3}, 2\right)$



$$A = 2\left[\int_0^2 \sqrt{6Y}\,dY + \int_2^4 \sqrt{16 - Y^2}\,dY\right]$$

$$= 2\left[\frac{2}{3}\sqrt{6}\left[Y^{3/2}\right]_0^2 + \left[\frac{1}{2}Y\sqrt{16 - Y^2} + \frac{16}{2}\sin^{-1}\frac{Y}{4}\right]_2^4\right] = \left(\frac{4\sqrt{3}}{3} + \frac{16\pi}{3}\right)\text{sq. units}$$

Do yourself : 3

(i) Find the area inside the circle $x^2-2x+y^2-4y+1=0$ and outside the ellipse $x^2-2x+4y^2-16y+13=0$

(b) If the equation of the curve is in parametric form, then $A=\int_{t=\alpha}^{t=\beta} y\frac{dx}{dt}.dt$ or $\int_{t=\gamma}^{t=\delta} x\frac{dy}{dt}.dt$, where α & β are values corresponding to values of x and γ & δ are values corresponding to values of y.

Illustration 12 : Find the area bounded by x-axis and the curve given by x = asint, y = acost for $0 \le t \le \pi$.

Solution :

$$\text{Area} = \int_0^{\pi} y\frac{dx}{dt}.dt = \int_0^{\pi} a\cos t(a\cos t)dt = \frac{a^2}{2}\int_0^{\pi}(1+\cos 2t)dt = \frac{a^2}{2}\left|t+\frac{\sin 2t}{2}\right|_0^{\pi} = \frac{a^2}{2}|\pi| = \frac{\pi a^2}{2}$$

Alternatively,

$$\text{Area} = \int_0^{\pi} x\frac{dy}{dt}.dt = \left|\int_0^{\pi} a\sin t(-a\sin t)dt\right| = \frac{a^2}{2}\left|\int_0^{\pi}(\cos 2t-1)dt\right| = \frac{a^2}{2}\left|-\frac{\sin 2t}{2}-t\right|_0^{\pi} = \frac{\pi a^2}{2}$$

Illustration 13 : Find the area of the figure bounded by one arc of the cycloid x = a(t − sint), y = a(1 − cost) and the x-axis.

Solution : To find the points where an arc cuts x-axis

$a(1-\cos t)=0 \quad \Rightarrow \quad t=0, \pi$

$$\text{Area} = \int_0^{\pi} y\frac{dx}{dt}dt = \int_0^{\pi} a^2(1-\cos t)^2 dt = a^2\left|\frac{3}{2}t-2\sin t+\frac{\sin 2t}{4}\right|_0^{\pi} = a^2\left[\frac{3\pi}{2}\right] = \frac{3\pi a^2}{2}$$

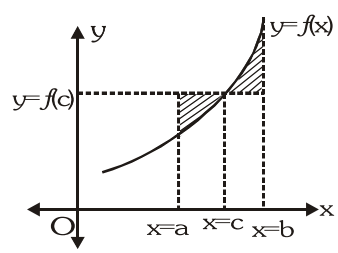
Do yourself - 4 :

(i) Find the area of the loop of the curve :

(a) $x=3t^2,\ y=3t-t^3$ (b) $x=t^2-1,\ y=t^3-t$

(c) If $y=f(x)$ is a monotonic function in (a, b), then the area bounded by the ordinates at x = a, x = b, $y=f(x)$ and $y=f(c)$ [where $c \in (a,b)$] is minimum when $c=\frac{a+b}{2}$.

Proof : Let the function $y=f(x)$ be monotonically increasing.



$$\text{Required area } A = \int_a^c [f(c)-f(x)]dx + \int_c^b [f(x)-f(c)]dx$$

For minimum area, $\frac{dA}{dc}=0$

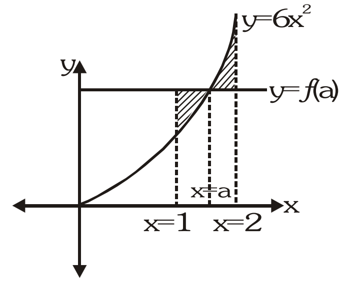
$$\Rightarrow \quad [f'(c).c+f(c)-f'(c)a-f(c)]+[-f(c)-f'(c).b+f'(c).c+f(c)]=0$$

$$\Rightarrow \quad f'(c)\left\{c-\frac{a+b}{2}\right\}=0$$

$$\Rightarrow \quad c=\frac{a+b}{2} \quad (\because f'(c)\ne 0)$$

Illustration 14 : Find the value of 'a' for which area bounded by x = 1, x=2, $y=6x^2$ and $y=f(a)$ is minimum.

Solution : Let $b = f(a)$.



$$A = \int_1^a (b-6x^2)dx + \int_a^2 (6x^2-b)dx = \left|bx-2x^3\right|_1^a + \left|2x^3-bx\right|_a^2$$

$= 8a^3 - 18a^2 + 18$

For minimum area $\frac{dA}{da} = 0$

$\Rightarrow \quad 24a^2 - 36a = 0 \quad \Rightarrow \quad a = 1.5$

Alternatively, $y = 6x^2 \quad \Rightarrow \quad \frac{dy}{dx} = 12x$

Hence $y = f(x)$ is monotonically increasing. Hence bounded area is minimum when

$a = \left(\frac{1+2}{2}\right) = 1.5$

Do yourself - 5 :

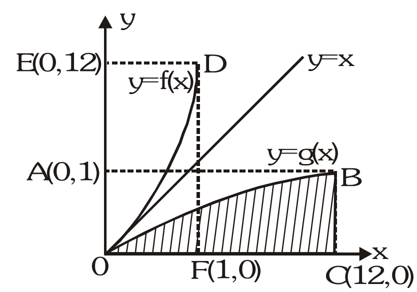
(i) Find the value of 'a' $(0 < a < \frac{\pi}{2})$ for which the area bounded by the curve $f(x) = \sin^3 x + \sin x$, $y = f(a)$ between $x = 0$ & $x = \pi$ is minimum.

(d) The area bounded by a curve & an axis is equal to the area bounded by the inverse of that curve & the other axis, i.e., the area bounded by $y = f(x)$ and x-axis (say) is equal to the area bounded by $y = f^{-1}(x)$ and y-axis.

Illustration 15 : If y = g(x) is the inverse of a bijective mapping $f : R \to R$, $f(x) = 6x^5 + 4x^3 + 2x$, find the area bounded by g(x), the x-axis and the ordinate at x = 12.

Solution : $f(x) = 12$

$\Rightarrow \quad 6x^5 + 4x^3 + 2x = 12 \quad \Rightarrow \quad x = 1$



$\int_0^{12} g(x)dx$ = area of rectangle OEDF $- \int_0^1 f(x)dx$

$= 1 \quad 12 - \int_0^1 (6x^5 + 4x^3 + 2x)dx = 12 - 3 = 9$ sq. units.

Do yourself - 6 :

(i) Find the area bounded by the inverse of bijective function $f(x) = 4x^3 + 6x$, the x-axis and the ordinates $x = 0$ & $x = 44$.

5. USEFUL RESULTS :

(a) Whole area of the ellipse, $x^2/a^2 + y^2/b^2 = 1$ is πab sq.units.

(b) Area enclosed between the parabolas $y^2 = 4ax$ & $x^2 = 4by$ is $16ab/3$ sq.units.

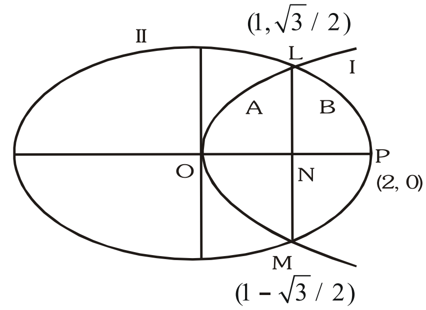
(c) Area included between the parabola $y^2 = 4ax$ & the line $y = mx$ is $8a^2/3m^3$ sq.units.

(d) The area of the region bounded by one arch of sin ax (or cosbx) and x-axis is $2/a$ sq.units.

(e) Average value of a function $y = f(x)$ over an interval $a \le x \le b$ is defined as : $y(av) = \frac{1}{b-a}\int_a^b f(x)dx$.

Miscellaneous Illustration :

Illustration 16 : Find the smaller of the areas bounded by the parabola $4y^2 - 3x - 8y + 7 = 0$ and the ellipse $x^2 + 4y^2 - 2x - 8y + 1 = 0$.



Solution : C_1 is $4(y^2 - 2y) = 3x - 7$

or $\quad 4(y-1)^2 = 3x - 3 = 3(x-1)$ .....(i)

Above is parabola with vertex at (1, 1)

C_2 is $(x^2 - 2x) + 4(y^2 - 2y) = -1$

or $\quad (x-1)^2 + 4(y-1)^2 = -1 + 1 + 4$

or $\quad \dfrac{(x-1)^2}{2^2} + \dfrac{(y-1)^2}{1^2} = 1$ ......(ii)

Above represents an ellipse with centre at (1, 1). Shift the origin to (1, 1) and this will not affect the magnitude of required area but will make the calculation simpler.

Thus the two curves are

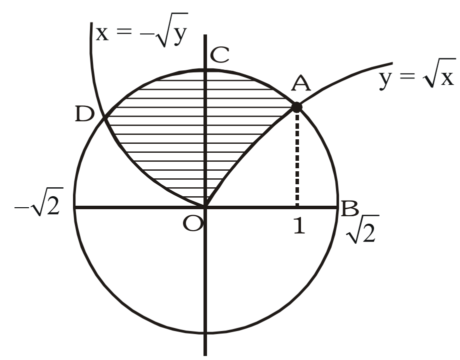
$4Y^2 = 3X$ and $\dfrac{X^2}{2^2} + \dfrac{Y^2}{1} = 1$

They meet at $\left(1, \pm\dfrac{\sqrt{3}}{2}\right)$

Required area $= 2(A + B) = 2\left[\int Y_1 dX + \int Y_2 dX\right]$

$$= 2\left[\frac{\sqrt{3}}{2}\int_0^1 \sqrt{X}\,dX + \int_1^2 \frac{\sqrt{4-X^2}}{2}dX\right] = \left[\frac{\sqrt{3}}{6} + \frac{2\pi}{3}\right] \text{ sq.units.}$$

Illustration 17 : Find the area bounded by the regions $y \geq \sqrt{x}$, $x > -\sqrt{y}$ & curve $x^2 + y^2 = 2$.



Solution : Common region is given by the diagram

If area of region OAB = λ

then area of OCD = λ

Because $y = \sqrt{x}$ & $x = -\sqrt{y}$

will bound same area with x & y axes respectively.

$y = \sqrt{x} \Rightarrow y^2 = x$

$x = -\sqrt{y} \Rightarrow x^2 = y$ and hence both the curves are symmetric with respect to the line $y = x$

Area of first quadrant OBC $= \dfrac{\pi r^2}{4} = \dfrac{\pi}{2}$ $\quad (\because \; r = \sqrt{2})$

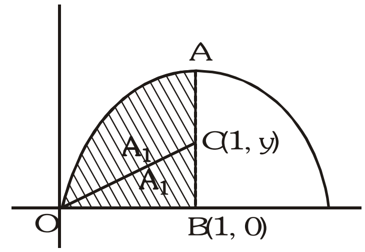
Area of region OCA $= \dfrac{\pi}{2} - \lambda$

Area of shaded region $= \left(\dfrac{\pi}{2} - \lambda\right) + \lambda = \dfrac{\pi}{2}$ sq.units.

Illustration 18 : Find the equation of line passing through the origin & dividing the curvilinear triangle with vertex at the origin, bounded by the curves $y = 2x - x^2$, $y = 0$ & $x = 1$ in two parts of equal areas.

Solution : Area of region OBA $= \int_0^1 (2x - x^2)dx$

$$= \left[x^2 - \frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$



$\frac{2}{3} = A_1 + A_1 \Rightarrow A_1 = \frac{1}{3}$

Let pt. C has coordinates (1, y)

Area of $\Delta OCB = \frac{1}{2}\ \ 1\ \ \ y = \frac{1}{3}$

$y = \frac{2}{3}$

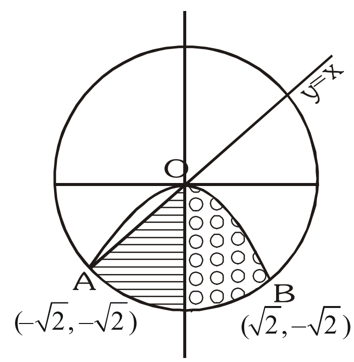
C has coordinates $\left(1, \frac{2}{3}\right)$

Line OC has slope $m = \dfrac{\frac{2}{3} - 0}{1 - 0} = \frac{2}{3}$

Equation of line OC is $y = mx \Rightarrow y = \frac{2}{3}x$.

Illustration 19 : Find the area bounded by the curves $x^2 + y^2 = 4$, $x^2 = -\sqrt{2}\,y$ and the line $x = y$, below x-axis.

Solution : Let C is $x^2 + y^2 = 4$, P is $y = -\frac{x^2}{\sqrt{2}}$ and L is $y = x$.



We have above three curves.

Solving P and C we get the points

$A(-\sqrt{2}\,, -\sqrt{2}\,)$, $B(\sqrt{2}\,, -\sqrt{2}\,)$

Also the line $y = x$ passes through $A(-\sqrt{2}\,, -\sqrt{2}\,)$

$\therefore$ Required area = shaded + dotted

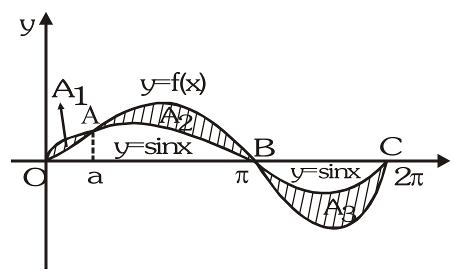
$$= \int_{-\sqrt{2}}^{0}(y_3 - y_1)dx + \int_0^{\sqrt{2}}(y_2 - y_1)dx$$

$$= \int_{-\sqrt{2}}^{0} x\,dx + \int_0^{\sqrt{2}} \frac{-x^2}{\sqrt{2}}dx - \int_{-\sqrt{2}}^{\sqrt{2}}\sqrt{4 - x^2}\,dx$$

$$= \left[\frac{x^2}{2}\right]_{-\sqrt{2}}^{0} - \frac{1}{\sqrt{2}}\left[\frac{x^3}{3}\right]_0^{\sqrt{2}} - \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{2}}$$

$\therefore \quad |A| = \dfrac{3\pi + 16}{6}$ sq.units.

Illustration 20 : In the adjacent figure, graphs of two functions $y = f(x)$ and $y = \sin x$ are given. $y = \sin x$ intersects $y = f(x)$ at A(a f(a)); B $(\pi, 0)$ and C$(2\pi, 0)$. A_i $(i = 1, 2, 3)$ is the area bounded by the curves $y = f(x)$ and $y = \sin x$ between $x = 0$ and $x = a$ $(i = 1)$, between $x = a$ and $x = \pi$ $(i = 2)$, between $x = \pi$ and $x = 2\pi$ $(i = 3)$. If $A_1 = 1 - \sin a + (a - 1)\cos a$, determine the function $f(x)$. Hence determine 'a' and A_1. Also calculate A_2 and A_3.



Solution : From the figure it is clear that $A_1 = \int_0^a (\sin x - f(x))\,dx = 1 - \sin a + (a-1)\cos a$

differentiate w.r.t. a

$\sin a - f(a) = -\cos a + \cos a - (a-1)\sin a$

$\sin a - f(a) = -a\sin a + \sin a$

$f(a) = a\sin a$

$\Rightarrow \quad f(x) = x\sin x$

The points where f(x) & sinx intersect are

$x\sin x = \sin x \Rightarrow \sin x = 0$ or $x = 1$

$a = 1 \ (0 < a < \pi)$

$A_1 = \int_0^1 (\sin x - x\sin x)\,dx = 1 - \sin 1$ sq.units

$A_2 = \int_1^{\pi} (f(x) - \sin x)\,dx = \int_1^{\pi} (x\sin x - \sin x)\,dx = (\pi - 1 - \sin 1)$ sq.units

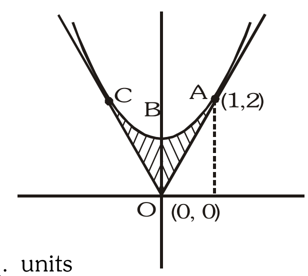
$A_3 = \left|\int_{\pi}^{2\pi} (\sin x - x\sin x)\,dx\right| = (3\pi - 2)$ sq.units

Illustration 21 : The area bounded by $y = x^2 + 1$ and the tangents to it drawn from the origin is :-

(A) 8/3 sq. units (B) 1/3 sq. units (C) 2/3 sq. units (D) none of these

Solution : The parabola is even function & let the equation of tangent is y= mx

Now we calculate the point of intersection of parabola & tangent



$mx = x^2 + 1$

$x^2 - mx + 1 = 0 \Rightarrow D = 0$

$\Rightarrow m^2 - 4 = 0 \Rightarrow m = \pm 2$

Two tangents are possible $y = 2x$ & $y = -2x$

Intersection of $y = x^2 + 1$ & $y = 2x$ is $x = 1$ & $y = 2$

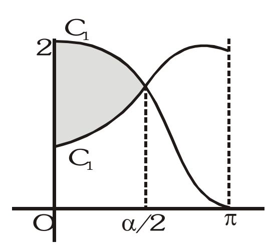
Area of shaded region OAB $= \int_0^1 (y_2 - y_1)\,dx = \int_0^1 ((x^2+1) - 2x)\,dx = \frac{1}{3}$ sq. units

Area of total shaded region $= 2\left(\frac{1}{3}\right) = \frac{2}{3}$ sq. units

Illustration 22 : **Determination of unknown parameter :**

Consider the two curves $C_1 : y = 1 + \cos x$ & $C_2 : y = 1 + \cos(x - \alpha)$ for $\alpha \in \left(0, \frac{\pi}{2}\right); x \in [0, \pi]$. Find the value of α, for which the area of the figure bounded by the curves C_1, C_2 & $x = 0$ is same as that of the figure bounded by C_2, $y = 1$ & $x = \pi$.

Solution : $1 + \cos x = 1 + \cos(x - \alpha) \Rightarrow x = \frac{\alpha}{2}$



$A_1 = \int_0^{\alpha/2} (1 + \cos x) - (1 + \cos(x - \alpha))dx$

$= \left|\sin x - \sin(x - \alpha)\right|_0^{\alpha/2} = 2\sin\frac{\alpha}{2} - \sin\alpha$

$1 + \cos(x - \alpha) = 1 \Rightarrow x = \alpha + \dfrac{\pi}{2}$

$$A_2 = \int_{\alpha+\frac{\pi}{2}}^{\pi} \left(1 + \cos\left(x - \alpha\right) - 1\right) dx = \left|\sin\left(x - \alpha\right)\right|_{\alpha+\frac{\pi}{2}}^{\pi}$$

$= |\sin \alpha - 1| = 1 - \sin \alpha$

$A_1 = A_2 \quad \Rightarrow \quad 2\sin\dfrac{\alpha}{2} - \sin\alpha = 1 - \sin\alpha$

$\alpha = \dfrac{\pi}{3}$ Ans.

ANSWERS FOR DO YOURSELF

1 :	(i)	$\frac{25}{3}$ sq. units	(ii)	$\frac{\pi}{4}$ sq. units.	(iii)	8 sq. units.	(iv)	$\frac{7}{24}$ sq. units
2 :	(i)	$\frac{1}{6}$ sq. units	(ii)	4 sq. units	(iii)	$\frac{4-\sqrt{2}}{\log 2} - \frac{5}{2}\log 2 + \frac{3}{2}$ sq. units		
3 :	(i)	2π sq. units						
4 :	(i)	(a) $\frac{72}{5}\sqrt{3}$ sq. units	(b)	$\frac{8}{15}$ sq. units				
5 :	(i)	$\frac{\pi}{8}$						
6 :	(i)	60 sq. units.						

EXERCISE - 01

CHECK YOUR GRASP

SELECT THE CORRECT ALTERNATIVE (ONLY ONE CORRECT ANSWER)

1. The area of the region bounded by the curves $y = |x - 2|$, $x = 1$, $x = 3$ and the x-axis is -
(A) 3 (B) 2 (C) 1 (D) 4

2. The area enclosed between the curve $y = \log_e(x + e)$ and the coordinate axes is -
(A) 4 (B) 3 (C) 2 (D) 1

3. The area of the figure bounded by the curves $y = \ell nx$ & $y = (\ell nx)^2$ is -
(A) $e + 1$ (B) $e - 1$ (C) $3 - e$ (D) 1

4. Suppose $y = f(x)$ and $y = g(x)$ are two functions whose grahps intersect at three points (0, 4), (2, 2) and (4, 0) with $f(x) > g(x)$ for $0 < x < 2$ and $f(x) < g(x)$ for $2 < x < 4$.

If $\int_0^4 [f(x) - g(x)]dx = 10$ and $\int_2^4 [g(x) - f(x)]dx = 5$, the area between two curves for $0 < x < 2$, is -
(A) 5 (B) 10 (C) 15 (D) 20

5. The area bounded by the curves $y = -\sqrt{-x}$ and $x = -\sqrt{-y}$ where $x, y \le 0$
(A) cannot be determined (B) is 1/3 (C) is 2/3
(D) is same as that of the figure bounded by the curves $y = \sqrt{-x}$; $x \le 0$ and $x = \sqrt{-y}$; $y \le 0$

6. The area of the closed figure bounded by $y = x$, $y = -x$ & the tangent to the curve $y = \sqrt{x^2 - 5}$ at the point (3, 2) is -
(A) 5 (B) $2\sqrt{5}$ (C) 10 (D) $\frac{5}{2}$

7. The area of the region(s) enclosed by the curves $y = x^2$ and $y = \sqrt{|x|}$ is -
(A) 1/3 (B) 2/3 (C) 1/6 (D) 1

8. The area bounded by the curve $y = xe^{-x}$; $xy = 0$ and $x = c$, where c is the x-coordinate of the curve's inflection point, is -
(A) $1 - 3e^{-2}$ (B) $1 - 2e^{-2}$ (C) $1 - e^{-2}$ (D) 1

9. The area enclosed by the curves $y = \cos x$, $y = 1 + \sin 2x$ and $x = \frac{3\pi}{2}$ as x varies from 0 to $\frac{3\pi}{2}$, is -
(A) $\frac{3\pi}{2} - 2$ (B) $\frac{3\pi}{2}$ (C) $2 + \frac{3\pi}{2}$ (D) $1 + \frac{3\pi}{2}$

10. The area enclosed by the curve $y^2 + x^4 = x^2$ is -
(A) $\frac{2}{3}$ (B) $\frac{4}{3}$ (C) $\frac{8}{3}$ (D) $\frac{10}{3}$

11. Consider two curves $C_1 : y = \frac{1}{x}$ and $C_2 : y = \ell nx$ on the xy plane. Let D_1 denotes the region surrounded by C_1, C_2 and the line $x = 1$ and D_2 denotes the region surrounded by C_1, C_2 and the line $x = a$. If $D_1 = D_2$ then the value of 'a' -
(A) $\frac{e}{2}$ (B) e (C) $e - 1$ (D) $2(e - 1)$

12. The area of the region for which $0 < y < 3 - 2x - x^2$ & $x > 0$ is -
(A) $\int_1^3 (3 - 2x - x^2)dx$ (B) $\int_0^3 (3 - 2x - x^2)dx$ (C) $\int_0^1 (3 - 2x - x^2)dx$ (D) $\int_1^3 (3 - 2x - x^2)dx$

13. The area bounded by the curves $y = x(1 - \ell nx)$ and positive x-axis between $x = e^{-1}$ and $x = e$ is -

(A) $\left(\frac{e^2-4e^{-2}}{5}\right)$ (B) $\left(\frac{e^2-5e^{-2}}{4}\right)$ (C) $\left(\frac{4e^2-e^{-2}}{5}\right)$ (D) $\left(\frac{5e^2-e^{-2}}{4}\right)$

14. The curve $f(x) = Ax^2 + Bx + C$ passes through the point (1, 3) and line $4x + y = 8$ is tangent to it at the point (2, 0). The area enclosed by $y = f(x)$, the tangent line and the y-axis is -

(A) 4/3 (B) 8/3 (C) 16/3 (D) 32/3

15. Let $y = g(x)$ be the inverse of a bijective mapping $f : R \to Rf(x) = 3x^3 + 2x$. The area bounded by graph of g(x), the x-axis and the ordinate at x = 5 is -

(A) $\frac{5}{4}$ (B) $\frac{7}{4}$ (C) $\frac{9}{4}$ (D) $\frac{13}{4}$

16. A function $y = f(x)$ satisfies the differential equation, $\frac{dy}{dx} - y = \cos x - \sin x$, with initial condition that y is bounded when $x \to \infty$. The area enclosed by $y = f(x)$, $y = \cos x$ and the y-axis in the 1st quadrant is-

(A) $\sqrt{2}-1$ (B) $\sqrt{2}$ (C) 1 (D) $\frac{1}{\sqrt{2}}$

<u>SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)</u>

17. Let 'a' be a positive constant number. Consider two curves $C_1 : y = e^x$, $C_2 : y = e^{a-x}$. Let S be the area of the part surrounding by C_1, C_2 and the y-axis, then -

(A) $\lim_{a\to-\infty} S = 1$ (B) $\lim_{a\to 0}\frac{S}{a^2} = \frac{1}{4}$

(C) Range of S is $[0,\infty)$ (D) S(a) is neither odd nor even

18. Area enclosed by the curve $y = \sin x$ between $x = 2n\pi$ to $x = 2(n+1)\pi$ is-

(A) $\int_0^{2\pi} \sin x\, dx$ (B) $2\int_0^{\pi} \sin x\, dx$ (C) $4\int_0^{\pi/2} \sin x\, dx$ (D) 4

19. If (a, 0) & (b,0) [a,b > 0] are the points where the curve $y = \sin 2x - \sqrt{3}\sin x$ cuts the positive x-axis first & second time, A & B are the areas bounded by the curve & positive x-axis between x=0 to x=a and x = a to x=b respectively, then -

(A) $4A + 8\cos a = 7$ (B) $AB = \frac{1}{16}$ (C) $4A + 4B + 14\cos b = 0$ (D) $B - A = 4\cos a$

20. For which of the following values of m, is the area of the region bounded by the curve $y = x - x^2$ and the line $y = mx$ equals to 9/2 ?

(A) –4 (B) –2 (C) 2 (D) 4

21. Let $f(x) = |x| - 2$ and $g(x) = |f(x)|$.
Now area bounded by x-axis and f(x) is A_1 and area bounded by x-axis and g(x) is A_2 then –

(A) $A_1 = 3$ (B) $A_1 = A_2$ (C) $A_2 = 4$ (D) $A_1 + A_2 = 8$

CHECK YOUR GRASP — ANSWER KEY — EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	D	C	C	B	A	B	A	C	B
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	B	C	B	B	D	A	A,B,C,D	B,C,D	A,B,C,D	B,D
Que.	21									
Ans.	B,C,D									

EXERCISE - 02

BRAIN TEASERS

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

1. If $C_1 \equiv y = \frac{1}{1+x^2}$ and $C_2 \equiv y = \frac{x^2}{2}$ are two curves lying in the XY plane. Then -

(A) area bounded by curve C_1 and $y = 0$ is π
(B) area bounded by C_1 and C_2 is $\frac{\pi}{2} - \frac{1}{3}$
(C) area bounded by C_1 and C_2 is $1 - \frac{\pi}{2}$
(D) area bounded by curve C_1 and x-axis is $\frac{\pi}{2}$

2. Area enclosed by the curves $y = \ell nx$; $y = \ell n|x|$; $y = |\ell nx|$ and $y = |\ell n|x||$ is equal to -
(A) 2 (B) 4 (C) 8 (D) cannot be determined

3. $y = f(x)$ is a function which satisfies-
(i) $f(0) = 0$ (ii) $f''(x) = f'(x)$ and (iii) $f'(0) = 1$
then the area bounded by the graph of $y = f(x)$, the lines $x = 0$, $x - 1 = 0$ and $y + 1 = 0$, is -
(A) e (B) e – 2 (C) e – 1 (D) e + 1

4. Let T be the triangle with vertices (0, 0), $(0, c^2)$ and (c, c^2) and let R be the region between $y = cx$ and $y = x^2$ where $c > 0$ then -

(A) Area (R) = $\frac{c^3}{6}$ (B) Area of R = $\frac{c^3}{3}$ (C) $\lim\limits_{c\to 0^+} \frac{\text{Area(T)}}{\text{Area(R)}} = 3$ (D) $\lim\limits_{c\to 0^+} \frac{\text{Area(T)}}{\text{Area(R)}} = \frac{3}{2}$

5. Suppose $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 5$ and $h(x) = (fog)(x)$. The area enclosed by the graph of the function $y = f(x)$ and the pair of tangents drawn to it from the origin, is -
(A) 8/3 (B) 16/3 (C) 32/3 (D) none

6. Let $f(x) = x^2 + 6x + 1$ and R denote the set of points (x, y) in the coordinate plane such that $f(x) + f(y) \le 0$ and $f(x) - f(y) \le 0$. The area of R is equal to -
(A) 16π (B) 12π (C) 8π (D) 4π

7. The value of 'a' ($a > 0$) for which the area bounded by the curves $y = \frac{x}{6} + \frac{1}{x^2}$, $y = 0$, $x = a$ and $x = 2a$ has the least value, is -
(A) 2 (B) $\sqrt{2}$ (C) $2^{1/3}$ (D) 1

8. Consider the following regions in the plane :
$R_1 = \{(x, y) : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$ and $R_2 = \{(x, y) : x^2 + y^2 \le 4/3\}$

The area of the region $R_1 \cap R_2$ can be expressed as $\frac{a\sqrt{3} + b\pi}{9}$, where a and b are integers, then -
(A) a = 3 (B) a = 1 (C) b = 1 (D) b = 3

9. The area of the region of the plane bounded by $(|x|, |y|) \le 1$ & $xy \le \frac{1}{2}$ is -

(A) less then $4\ell n\ 3$ (B) $\frac{15}{4}$ (C) $2 + 2\ell n2$ (D) $3 + \ell n2$

10. The line $y = mx$ bisects the area enclosed by the curve $y = 1 + 4x - x^2$ & the lines $x = 0$, $x = \frac{3}{2}$ & $y = 0$. Then the value of m is -

(A) $\frac{13}{6}$ (B) $\frac{6}{13}$ (C) $\frac{3}{2}$ (D) 4

11. Area of the region enclosed between the curves $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$ is -
(A) 1 (B) 4/3 (C) 2/3 (D) 2

12. If the tangent to the curve $y = 1 - x^2$ at $x = \alpha$, where $0 < \alpha < 1$, meets the axes at P and Q. As α varies, the minimum value of the area of the triangle OPQ is k times the area bounded by the axes and the part of the curve for which $0 < x < 1$, then k is equal to -

(A) $\frac{2}{\sqrt{3}}$ (B) $\frac{75}{16}$ (C) $\frac{25}{18}$ (D) $\frac{2}{3}$

BRAIN TEASERS — ANSWER KEY — EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10	11	12			
Ans.	A,B	B	C	A,C	B	C	D	A,C	A,D	A	D	A			

EXERCISE - 03 MISCELLANEOUS TYPE QUESTIONS

MATCH THE COLUMN

Following questions contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

1. Let $f(x) = |x|$, $g(x) = |x - 1|$ and $h(x) = |x + 1|$.

	Column-I		Column-II
(A)	Area bounded by min (f(x), g(x)) and x-axis is	(p)	$\frac{1}{8}$ sq. unit
(B)	Area bounded by min (f(x), h(x)) and x-axis is	(q)	$\frac{1}{4}$ sq. unit
(C)	Area bounded by min ((f(x), g(x), h(x)) and x-axis is	(r)	$\frac{1}{2}$ sq. unit
(D)	Area bounded by min (f(x), g(x), h(x)) and $y = \frac{1}{2}$ is	(s)	$\frac{3}{4}$ sq. unit

2.

	Column I		Column – II
(A)	The area bounded by the curve $x = 3y^2 - 9$ and the lines $x = 0$, $y = 0$ and $y = 1$ in square units is equal to	(p)	1
(B)	If a curve $f(x) = a\sqrt{x} + bx$, ($f(x) \ge 0\ \forall\ x \in [0, 9]$) passes through the point (1, 2) and the area bounded by the curve, line $x = 4$ and x-axis is 8 square unit, then $2a + b$ is equal to	(q)	4
(C)	The area enclosed between the curves $y = \sin^2 x$ and $y = \cos^2 x$ in the interval $0 \le x \le \pi$ in square units in equal to	(r)	8
(D)	The area bounded by the curve $y^2 = 16x$ and line $y = mx$ is $\frac{2}{3}$ square units, then m is equal to	(s)	5

ASSERTION & REASON

These questions contains, Statement I (assertion) and Statement II (reason).

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.

(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I.

(C) Statement-I is true, Statement-II is false.

(D) Statement-I is false, Statement-II is true.

1. **Statement-I** : The area of the curve $y = \sin^2 x$ from 0 to π will be more than that of curve $y = \sin x$ from 0 to π.

Because

Statement-II : $t^2 > t$ if $t \in R - [0, 1]$.

(A) A (B) B (C) C (D) D

2. **Statement-I** : The area bounded by the curves $y = x^2 - 3$ and $y = kx + 2$ is the least, if $k = 0$.

Because

Statement-II : The area bounded by the curves $y = x^2 - 3$ and $y = kx + 2$ is $\sqrt{k^2 + 20}$.

(A) A (B) B (C) C (D) D

3. Consider the two curves $y = x - \lambda x^2$ and $y = \frac{x^2}{\lambda}$, ($\lambda > 0$).

Statement-I : The area bounded between the curves is maximum when $\lambda = 1$.

Because

Statement-II : The area bounded between the curves is $\frac{\lambda^2}{(1+\lambda^2)^2}$ square units.

(A) A (B) B (C) C (D) D

COMPREHENSION BASED QUESTIONS

Comprehension # 1

Consider two curves $y = \frac{1}{x^2}$ and $y = \frac{1}{4(x-1)}$. 'a' is a number such that a > 2 & the reciprocal of the area of the figure bounded by the curves, the line x = 2 & x = a is a itself. 'b' is a number such that 1 < b < 2 & the area bounded by the two curves & the lines x = b & x = 2 is equal to $1 - \frac{1}{b}$.

On the basis of above information, answer the following questions :

1. The value of $\ell na - \ell nb$ is -

(A) positive integer (B) negative integer

(C) rational number of the form $\frac{p}{q}$, where p, q are co-prime & q > 1. (D) irrational number

2. If $A = \begin{bmatrix} \ell n(a-1) & 0 \\ 0 & \ell n(b-1) \end{bmatrix}$, then A^{-1} is -

(A) $-\frac{A}{4}$ (B) A (C) 4A (D) $\frac{A}{4}$

3. If z is a complex number such that $z = \ell n(a-1) + i\ell n(b-1)$ then arg(z) is -

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$ (C) $\frac{-\pi}{4}$ (D) $\frac{-3\pi}{4}$

Comprehension # 2

Five curves defined as follows :

$C_1 : |x + y| \le 1$
$C_2 : |x - y| \le 1$
$C_3 : |x| \le \frac{1}{2}$
$C_4 : |y| \le \frac{1}{2}$
$C_5 : 3x^2 + 3y^2 = 1$

On the basis of above information, answer the following questions :

1. The area bounded by C_1 and C_2 which does not contain the area bounded by C_5, is -

(A) $2 - \frac{\pi}{4}$ (B) $2 - \frac{\pi}{6}$ (C) $2 - \frac{\pi}{3}$ (D) 2

2. That part of area of curve C_5 which does not contain points satisfying C_3 and C_4, is -

(A) $\frac{\pi}{3} - \frac{1}{2}$ (B) $\frac{\pi}{3} - 1$ (C) $\frac{\pi}{3} - \frac{1}{6}$ (D) $\frac{2\pi}{9} - \frac{1}{\sqrt{3}}$

3. That part of area which is bounded by C_1 and C_2 but not bounded by C_3 and C_4, is -

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) none of these

MISCELLANEOUS TYPE QUESTION	ANSWER KEY	EXERCISE-3

- ***Match the Column***

1. (A) → (q), (B) → (q), (C) → (r), (D) → (s) 2. (A) → (r), (B) → (s), (C) → (p), (D) → (q)

- ***Assertion & Reason***

1. D 2. C 3. C

- ***Comprehension Based Quesions***

Comprehension # 1 : 1. A 2. D 3. C
Comprehension # 2 : 1. C 2. D 3. A

EXERCISE - 04 [A] CONCEPTUAL SUBJECTIVE EXERCISE

1. Find the area bounded by $y = \cos^{-1}x$, $y = \sin^{-1}x$ and y-axis.
2. Find the value of c for which the area of the figure bounded by the curves $y = \sin 2x$, the straight lines $x = \pi/6$, $x = c$ & the abscissa axis is equal to 1/2.
3. Find the area of the region $\{(x, y) : 0 \le y \le x + 1,\ 0 \le y \le x + 1,\ 0 \le x \le 2\}$.
4. A figure is bounded by the curves $y = \left|\sqrt{2}\sin\dfrac{\pi x}{4}\right|$, $y = 0$, $x = 2$ & $x = 4$. At what angles to the positive x-axis straight lines must be drawn through (4 , 0) so that these lines divide the figure into three parts of the same size.
5. Find the area bounded by the curves $y = \sqrt{1-x^2}$ and $y = x^3 - x$. Also find the ratio in which the y-axis divideds this area.
6. Find the area bounded on the right by the line $x + y = 2$, on the left by the parabola $y = x^2$ and below by the x- axis.
7. The tangent to the parabola $y = x^2$ has been drawn so that the abscissa x_0 of the point of tangency belongs to the interval [1, 2]. Find x_0 for which the triangle bounded by the tangent, x-axis & the straight line $y = x_0^2$ has the greatest area.
8. Find the area of the region bounded by the curves , $y = \log_e x$, $y = \sin^4 \pi x$ & $x = 0$.
9. For what value of 'a' is the area bounded by the curve $y = a^2x^2 + ax + 1$ and the straight lines $y = 0$, $x = 0$ & $x = 1$ the least ?
10. The line $3x + 2y = 13$ divides the area enclosed by the curve, $9x^2 + 4y^2 - 18x - 16y - 11 = 0$ in two parts. Find the ratio of the larger area to the smaller area.
11. Find the values of m ($m > 0$) for which the area bounded by the line $y = mx + 2$ and the curve $x = 2y - y^2$ is ,
(i) 9/2 square units & (ii) minimum. Also find the minimum area.
12. Let $f(x) = \text{Maximum } \{x^2, (1 - x)^2, 2x(1 - x)\}$, where $0 \le x \le 1$. Determine the area of the region bounded by the curves $y = f(x)$, x-axis, $x = 0$ & $x = 1$.
13. If the area enclosed by the parabolas $y = a - x^2$ and $y = x^2$ is $18\sqrt{2}$ sq. units. Find the value of 'a'.
14. Find the area enclosed between the curves : $y = \log_e(x + e)$, $x = \log_e(1/y)$ & the x-axis.
15. Find the value(s) of the parameter 'a' ($a > 0$) for each of which the area of the figure bounded by the straight line $y = \dfrac{a^2 - ax}{1 + a^4}$ & the parabola $y = \dfrac{x^2 + 2ax + 3a^2}{1 + a^4}$ is the greatest.
16. Find the positive value of 'a' for which the parabola $y = x^2 + 1$ bisects the area of the rectangle with vertices (0, 0), (a, 0), $(0, a^2 + 1)$ and $(a, a^2 + 1)$.
17. Find the area bounded by $y = x + \sin x$ and its inverse between $x = 0$ and $x = 2\pi$.

CONCEPTUAL SUBJECTIVE EXERCISE — ANSWER KEY — EXERCISE-4(A)

1. $\left(2-\sqrt{2}\right)$ sq.units 2. $c = -\dfrac{\pi}{6}$ or $\dfrac{\pi}{3}$ 3. 23/6 sq. units 4. $\pi - \tan^{-1}\dfrac{2\sqrt{2}}{3\pi}$; $\pi - \tan^{-1}\dfrac{4\sqrt{2}}{3\pi}$

5. $\dfrac{\pi}{2}$ sq. units ; $\dfrac{\pi-1}{\pi+1}$ 6. 5/6 sq. units 7. $x_0 = 2$, $A(x_0) = 8$ 8. $\dfrac{11}{8}$ sq. units 9. $a = -3/4$

10. $\dfrac{3\pi + 2}{\pi - 2}$ 11. (i) $m = 1$, (ii) $m = \infty$; $A_{min} = 4/3$ sq.units 12. $17/27$ 13. $a = 9$ 14. 2 sq. units

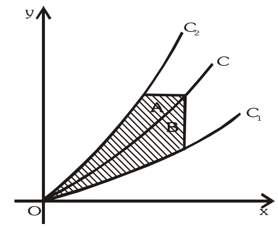
15. $a = 3^{1/4}$ 16. $\sqrt{3}$ 17. 8

EXERCISE - 04 [B] BRAIN STORMING SUBJECTIVE EXERCISE

1. For what value of 'a' is the area of the figure bounded by

$y = \frac{1}{x}$, $y = \frac{1}{2x-1}$, $x = 2$ & $x = a$ equal to $\ell n \frac{4}{\sqrt{5}}$?

2. Let C_1 & C_2 be two curves passing through the origin as shown in the figure. A curve C is said to "bisect the area" the region between C_1 & C_2, if for each point P of C, the two shaded regions A & B shown in the figure have equal areas. Determine the upper curve C_2, given that the bisecting curve C has the equation $y = x^2$ & that the lower curve C_1 has the equation $y = x^2/2$.



3. Let A_n be the area bounded by the curve $y = (\tan x)^n$ & the lines $x = 0$, $y = 0$ & $x = \pi/4$. Prove that for $n > 2$, $A_n + A_{n-2} = 1/(n-1)$ & deduce that $1/(2n+2) < A_n < 1/(2n-2)$.

4. Consider a square with vertices at (1, 1), (–1, 1), (–1, –1) & (1, –1). Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region S & find its area.

5. In what ratio does the x-axis divide the area of the region bounded by the parabolas $y = 4x - x^2$ and $y = x^2 - x$?

6. A polynomial function f(x) satisfies the condition $f(x + 1) = f(x) + 2x + 1$. Find f(x) if $f(0) = 1$. Find also the equations of the pair of tangents from the origin on the curve $y = f(x)$ and compute the area enclosed by the curve and the pair of tangents.

7. Consider the curve $y = x^n$ where $n > 1$ in the 1^{st} quadrant. If the area bounded by the curve, the x-axis and the tangent line to the graph of $y = x^n$ at the point (1, 1) is maximum then find the value of n.

8. Consider the collection of all curve of the form $y = a - bx^2$ that pass through the point (2, 1), where a and b are positive constants. Determine the value of a and b that will minimise the area of the region bounded by $y = a - bx^2$ and x-axis. Also find the minimum area.

9. Show that the area bounded by the curve $y = \frac{\ell nx - c}{x}$, the x-axis and the vertical line through the maximum point of the curve is independent of the constant c. Also find the area.

10. Let f(x) be a continuous function given by $f(x) = \begin{cases} 2x & \text{for } |x| \le 1 \\ x^2 + ax + b & \text{for } |x| > 1 \end{cases}$. Find the area of the region in the third quadrant bounded by the curves, $x = -2y^2$ and $y = f(x)$ lying on the left of the line $8x + 1 = 0$

[JEE 99, 10M (out of 200)]

BRAIN STORMING SUBJECTIVE EXERCISE — **ANSWER KEY** — EXERCISE-4(B)

1. $a = 8$ or $\frac{2}{5}\left(6-\sqrt{21}\right)$		**2.** $(16/9)\,x^2$	**4.** $\frac{1}{3}\left(16\sqrt{2}-20\right)$ sq. units
5. 4 : 121	**6.** $f(x) = x^2 + 1$; $y = \pm 2x$; $A = \frac{2}{3}$ sq. units		**7.** $\sqrt{2}+1$
8. $b = 1/8$, $A_{minimum} = 4\sqrt{3}$ sq. units		**9.** 1/2 sq. units	**10.** 257/192 ; $a = 2$; $b = -1$

EXERCISE - 05 [A]

JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

1. If the area bounded by the x-axis, curve y = f(x) and the lines x = 1, x = b is equal to $\sqrt{b^2+1} - \sqrt{2}$ for all b > 1, then f(x) is- [AIEEE-2002]

(1) $\sqrt{(x-1)}$ (2) $\sqrt{(x+1)}$ (3) $\sqrt{(x^2+1)}$ (4) $\frac{x}{\sqrt{1+x^2}}$

2. The area of the region bounded by the curves y = |x – 1| and y = 3 – |x| is - [AIEEE-2003]

(1) 6 sq. units (2) 2 sq. units (3) 3 sq. units (4) 4 sq. units

3. The area of the region bounded by the curves y = |x – 2|, x = 1, x = 3 and the x-axis is- [AIEEE-2004]

(1) 1 (2) 2 (3) 3 (4) 4

4. Area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is - [AIEEE-2005]

(1) 2ab (2) ab (3) $\sqrt{ab}$ (4) $\frac{a}{b}$

5. The area enclosed between the curve $y = \log_e(x+e)$ and the cooordinate axes is- [AIEEE-2005]

(1) 1 (2) 2 (3) 3 (4) 4

6. The parabolas $y^2 = 4x$ and $x^2 = 4y$ divide the square region bounded by the lines x = 4, y = 4 and the coordinate axes. If S_1, S_2, S_3 are respectively the areas of these parts numbered from top to bottom; then $S_1 : S_2 : S_3$ is - [AIEEE-2005]

(1) 1 : 2 : 1 (2) 1 : 2 : 3 (3) 2 : 1 : 2 (4) 1 : 1 : 1

7. Let f(x) be a non-negative continuous function such that the area bounded by the curve y= f(x), x-axis and the ordinates $x = \frac{\pi}{4}$ and $x = \beta > \frac{\pi}{4}$ is $\left(\beta\sin\beta+\frac{\pi}{4}\cos\beta+\sqrt{2}\beta\right)$. Then $f\left(\frac{\pi}{2}\right)$ is - [AIEEE-2005]

(1) $\left(\frac{\pi}{4}+\sqrt{2}-1\right)$ (2) $\left(\frac{\pi}{4}-\sqrt{2}+1\right)$ (3) $\left(1-\frac{\pi}{4}-\sqrt{2}\right)$ (4) $\left(1-\frac{\pi}{4}+\sqrt{2}\right)$

8. The area enclosed between the curves $y^2 = x$ and y = |x| is- [AIEEE-2007]

(1) $\frac{2}{3}$ (2) 1 (3) $\frac{1}{6}$ (4) $\frac{1}{3}$

9. The area of the plane region bounded by the curves $x + 2y^2 = 0$ and $x + 3y^2 = 1$ is equal to- [AIEEE-2008]

(1) $\frac{5}{3}$ (2) $\frac{1}{3}$ (3) $\frac{2}{3}$ (4) $\frac{4}{3}$

10. The area of the region bounded by the parabola $(y - 2)^2 = x - 1$, the tangent to the parabola at the point (2, 3) and the x–axis is :- [AIEEE-2009]

(1) 9 (2) 12 (3) 3 (4) 6

11. The area bounded by the curves y = cos x and y = sin x between the ordinates x = 0 and $x = \frac{3\pi}{2}$ is - [AIEEE-2010]

(1) $4\sqrt{2} - 2$ (2) $4\sqrt{2} + 2$ (3) $4\sqrt{2} - 1$ (4) $4\sqrt{2} + 1$

12. The area of the region enclosed by the curves $y = x$, $x = e$, $y = \frac{1}{x}$ and the positive x-axis is - [AIEEE-2011]

(1) $\frac{3}{2}$ square units (2) $\frac{5}{2}$ square units (3) $\frac{1}{2}$ square units (4) 1 square units

13. The area bounded by the curves $y^2 = 4x$ and $x^2=4y$ is:= [AIEEE-2011]

(1) 0 (2) $\frac{32}{3}$ (3) $\frac{16}{3}$ (4) $\frac{8}{3}$

14. The area bounded between the parabolas $x^2 = \frac{y}{4}$ and $x^2 = 9y$, and the straight line $y = 2$ is : [AIEEE-2012]

(1) $10\sqrt{2}$ (2) $20\sqrt{2}$ (3) $\frac{10\sqrt{2}}{3}$ (4) $\frac{20\sqrt{2}}{3}$

15. The area (in square units) bounded by the curves $y = \sqrt{x}$, $2y - x + 3 = 0$, x-axis and lying in the first quadrant is : [JEE (Main)-2013]

(1) 9 (2) 36 (3) 18 (4) $\frac{27}{4}$

PREVIOUS YEARS QUESTIONS **ANSWER KEY** EXERCISE-5 [A]

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans	4	4	1	1	1	4	4	3	4	1	1	1	3	4	1

EXERCISE - 05 [B] JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

1. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1,1) and the coordinates axes, lies in the first quadrant. If its area is 2, then the value of b is - [JEE 2001]

(A) –1 (B) 3 (C) –3 (D) 1

2. The area bounded by the curves $y = |x| - 1$ and $y = -|x| + 1$ is - [JEE 2002 (Screening), 3M]

(A) 1 (B) 2 (C) $2\sqrt{2}$ (D) 4

3. Find the area of the region bounded by the curves $y = x^2$, $y = |2 - x^2|$ and $y = 2$, which lies to the right of the line $x = 1$. [JEE 2002, (Mains) 5M out of 60]

4. (a) The area of the quadrilateral formed by the tangents at the end points of latus recta to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$, is -

(A) 27/4 sq. units (B) 9 sq. units (C) 27 sq. units (D) 27/2 sq. units

(b) The area bounded by the curves $y = \sqrt{x}$, $2y + 3 = x$ and x-axis in the 1^{st} quadrant is -

(A) 18 (B) 27/4 (C) 36 (D) 9

[JEE 2003 (Screening), 3 + 3M]

5. (a) The area bounded by the angle bisectors of the lines $x^2 - y^2 + 2y = 1$ and the line $x + y = 3$, is -

(A) 2 (B) 3 (C) 4 (D) 6

(b) The area enclosed between the curves $y = ax^2$ and $x = ay^2$ $(a > 0)$ is 1 sq. unit, then the value of a is -

(A) $\frac{1}{\sqrt{3}}$ (B) $\frac{1}{2}$ (C) 1 (D) $\frac{1}{3}$

[JEE 2004 (Screening), 3+3M]

6. The area bounded by the parabolas $y = (x + 1)^2$ and $y = (x - 1)^2$ and the line $y = 1/4$ is -

(A) 4 sq. units (B) 1/6 sq. units (C) 4/3 sq. units (D) 1/3 sq. units

[JEE 2005 (Screening), 3M]

7. Find the area bounded by the curves $x^2 = y$, $x^2 = -y$ and $y^2 = 4x - 3$. [JEE 2005, (Mains), 4M]

8. If $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}$, f(x) is a quadratic function and its maximum value occurs at a point V. A is a point of intersection of $y = f(x)$ with x-axis and point B is such that chord AB subtends a right angle at V. Find the area enclosed by f(x) and chord AB. [JEE 2005 (Mains) 6M out of 60]

9. Match the following -

(A)	$\int_0^{\pi/2} (\sin x)^{\cos x} (\cos x \cot x - \ell n(\sin x)^{\sin x}) dx$	(p)	1
(B)	Area bounded by $-4y^2 = x$ and $x - 1 = -5y^2$	(q)	0
(C)	Cosine of the angle of intersection of curves $y = 3^{x-1} \ln x$ and $y = x^x - 1$ is	(r)	$6 \ell n 2$
(D)	Let $\frac{dy}{dx} = \frac{6}{x+y}$, where y (0) = 0, then the value of y when $x + y = 6$ is	(s)	4/3

[JEE 2006, 6M]

Paragraph for Question Nos. 10 to 12

Consider the functions defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line. If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real valued differentiable function $y = f(x)$. If $x \in (-2, 2)$, the equation implicitly defines a unique real valued differentiable function $y = g(x)$ satisfying $g(0) = 0$.

10. If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2}) =$ **[JEE 2008, (4M, –1M)]**

(A) $\frac{4\sqrt{2}}{7^3 3^2}$ (B) $-\frac{4\sqrt{2}}{7^3 3^2}$ (C) $\frac{4\sqrt{2}}{7^3 3}$ (D) $-\frac{4\sqrt{2}}{7^3 3}$

11. The area of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$, is - **[JEE 2008, (4M, –1M)]**

(A) $\int_a^b \frac{x}{3((f(x))^2 - 1)} dx + bf(b) - af(a)$ (B) $-\int_a^b \frac{x}{3\left((f(x))^2 - 1\right)} dx + bf(b) - af(a)$

(C) $\int_a^b \frac{x}{3\left((f(x))^2 - 1\right)} dx - bf(b) + af(a)$ (D) $-\int_a^b \frac{x}{3\left((f(x))^2 - 1\right)} dx - bf(b) + af(a)$

12. $\int_{-1}^{1} g'(x)\, dx =$ **[JEE 2008, (4M, –1M)]**

(A) $2g(-1)$ (B) 0 (C) $-2g(1)$ (D) $2g(1)$

13. The area of the region between the curves $y = \sqrt{\frac{1 + \sin x}{\cos x}}$ and $y = \sqrt{\frac{1 - \sin x}{\cos x}}$ bounded by the lines $x = 0$ and $x = \frac{\pi}{4}$ is :- **[JEE 2008, (3M, –1M)]**

(A) $\int_0^{\sqrt{2}-1} \frac{t}{(1 + t^2)\sqrt{1 - t^2}} dt$ (B) $\int_0^{\sqrt{2}-1} \frac{4t}{(1 + t^2)\sqrt{1 - t^2}} dt$

(C) $\int_0^{\sqrt{2}+1} \frac{4t}{(1 + t^2)\sqrt{1 - t^2}} dt$ (D) $\int_0^{\sqrt{2}+1} \frac{t}{(1 + t^2)\sqrt{1 - t^2}} dt$

14. Area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is - **[JEE 2009, (4M, –1M)]**

(A) $e - 1$ (B) $\int_1^e \ln(e + 1 - y) dy$ (C) $e - \int_0^1 e^x dx$ (D) $\int_1^e \ln y\, dy$

Paragraph for Question 15 to 17 **[JEE 10, (3M each), –1M]**

Consider the polynomial

$f(x) = 1 + 2x + 3x^2 + 4x^3$.

Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$.

15. The real number s lies in the interval

(A) $\left(-\frac{1}{4}, 0\right)$ (B) $\left(-11, -\frac{3}{4}\right)$ (C) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (D) $\left(0, \frac{1}{4}\right)$

16. The area bounded by the curve $y = f(x)$ and the lines $x = 0$, $y = 0$ and $x = t$, lies in the interval

(A) $\left(\frac{3}{4}, 3\right)$ (B) $\left(\frac{21}{64}, \frac{11}{16}\right)$ (C) $(9, 10)$ (D) $\left(0, \frac{21}{64}\right)$

17. The function f'(x) is

(A) increasing in $\left(-t, -\frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$

(B) decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$

(C) increasing in $(-t, t)$

(D) decreasing in $(-t, t)$

18. **(a)** Let the straight line x = b divide the area enclosed by $y = (1 - x)^2$, y = 0 and x = 0 into two parts $R_1(0 \le x \le b)$ d and $R_2(b \le x \le 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then b equals

(A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

(b) Let $f:[-1,2] \to [0,\infty)$ be a continuous function such that $f(x) = f(1-x)$ for all $x \in [-1,2]$. Let $R_1 = \int_{-1}^{2} x f(x)dx$, and R_2 be the area of the region bounded by $y = f(x)$, x=–1, x=2, and the x-axis. Then -

(A) $R_1 = 2R_2$ (B) $R_1 = 3R_2$ (C) $2R_1 = R_2$ (D) $3R_1 = R_2$

[JEE 2011, 3+3M]

19. The area enclosed by the curve y = sinx + cosx and y = |cosx – sinx| over the interval $\left[0, \frac{\pi}{2}\right]$ is

[JEE(Advanced) 2013, 2M]

(A) $4(\sqrt{2}-1)$ (B) $2\sqrt{2}(\sqrt{2}-1)$ (C) $2(\sqrt{2}+1)$ (D) $2\sqrt{2}(\sqrt{2}+1)$

PREVIOUS YEARS QUESTIONS **ANSWER KEY** EXERCISE-5 [B]

1. C **2.** B **3.** $\left(\frac{20}{3} - 4\sqrt{2}\right)$ sq. units **4.** (a) C; (b) D **5.** (a) A; (b) A **6.** D

7. $\frac{1}{3}$ sq. units **8.** $\frac{125}{3}$ sq. units **9.** (A) → (p), (B) → (s), (C) → (p), (D) → (r) **10.** B

11. A **12.** D **13.** B **14.** B, C, D **15.** C **16.** A **17.** B

18. (a) B; (b) C **19.** B